Complex Analysis 2008 (ii)

(a) (i) derivative of f at $z_0 \in \mathbb{C}$:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

(ii) differentiable at every point in Ω.

(b) $f(z)$ is holomorphic in $D(0,1)$.

Prove $g(z) = \overline{f}(\overline{z})$ is too.

$f(z)$ hol: $\forall z_0 \in D \ \exists \ f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$.

$$\text{so} \quad \frac{g(z_0+h) - g(z_0)}{h} = \frac{\overline{f(\overline{z_0+h}) - f(\overline{z_0})}}{h}$$

$H = \overline{h} \Rightarrow$

$$= \overline{\left[\frac{f(\overline{z_0}+H) - f(\overline{z_0})}{H}\right]} = \overline{\left[\frac{f(w+H) - f(w)}{H}\right]}$$

$\overline{z_0} = w$

$$\text{Take} \lim_{H \to 0} = \overline{f'(w)}.$$

(c) (i) Ratio test: (i)

$$\lim_{k \to \infty} \frac{|(k+1)^{1/3} 2^{-k-1} (z-1)^{k+1}|}{|k^{1/3} 2^{-k} (z-1)^k|}$$

$$\lim_{k \to \infty} \left[\frac{(k+1)}{k}\right]^{1/3} \frac{1}{2} |z-1| \qquad \text{~~this so if z~~}$$

$$= \lim \left[1 + \frac{1}{k}\right]^{1/3} \frac{1}{2} |z-1|$$

$$= \frac{1}{2}|z-1| \qquad < 1 \text{ if } |z-1| < 2$$

$\Rightarrow$ RoC is 2.

(ii)

$$\lim_{n\to\infty}\left|\frac{(n+1)!\,(z-e)^{3n+3}}{n!\,(z-e)^{3n}}\right| = \left|(n+1)(z-e)^3\right| \longrightarrow \infty$$

$$\Rightarrow \text{RoC} = 0.$$

(iii)

$$\lim\left|\frac{z^{k+1}\,(k!)^2}{z^k\,\cancel{(k!)^2}\,((k+1)!)^2}\right| = \lim\left|\frac{z}{(k+1)^2}\right| \longrightarrow 0 \quad \text{as } k\to\infty$$

$$\Rightarrow \text{RoC} = \infty.$$

2. (a) CRE: $u_x = v_y$

$u_y = -v_x$

(b) $f(z) = u(x,y) + iv(x,y)$ is holomorphic on D

$\Rightarrow$ CRE apply.

$|f(z)| = \text{const} \Rightarrow u^2 + v^2 = c$

$\frac{\partial}{\partial x}$: $\cancel{2}uu_x + \cancel{2}vv_x = 0$

$\frac{\partial}{\partial y}$: $uu_y + vv_y = 0$

ie. $uu_x - vu_y = 0$

$uu_y + vu_x = 0$

ie. $vuu_x - v^2u_y = 0$

$u^2u_y + vuu_x = 0$ $\Rightarrow (\overset{c}{\overbrace{u^2+v^2}})u_y = 0.$

Similarly $(u^2+v^2)u_x = cu_x = 0$

$\vdots$

$\Rightarrow u_x = u_y = v_x = v_y = 0 \Rightarrow$ $f(z)$ is constant.

(c) (i) $\sin z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}(-1)^k$

$\cos z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}(-1)^k$

(ii) $\cos z = \cos(z - z_0 + z_0)$

$= \cos(z-z_0)\cos z_0 - \sin(z-z_0)\sin z_0$

$= \cos z_0 \sum_{k=0}^{\infty} \frac{(z-z_0)^{2k}(-1)^k}{2k!} - \sin z_0 \sum_{k=0}^{\infty} \frac{(z-z_0)^{2k+1}}{(2k+1)!}(-1)^k$

(iii) Differentiability of power series says that the derivative of a power series is another power series with the same RoC, and you can diff'te term-by-term

(simple)

(iv) $f(z) = \sin^2 z + \cos^2 z$

$f'(z) = 2\sin z\cos z - 2\cos z\sin z = 0$

$\Rightarrow f(z) = \text{const}.$

$f(0) = 1 \Rightarrow f(z) = 1 \ \forall z \in \mathbb{C}.$

3 (a) $f = u + iv$

$g = v + iu$

analytic $\Rightarrow$ CRE hold

~~ie. $u_x = v_y$, $u_y = -v_x$~~

~~f_x: $u_x + iv_x = v_y - iu_y$~~

~~g_x: $v_x + iu_x = -u_y + iv_y$~~

ie. for f: $u_x = v_y$, $u_y = -v_x$

for g: $v_x = u_y$, $v_y = -u_x$

$u_x = u_y = v_x = v_y = 0$.

$\Rightarrow u, v$ const f^{n}.

(b) $$c_k = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w^{k+1}}\,dw,$$ Γ is a positively oriented circular contour centred at $z_0 = 0$ and of radius $r \in (0, R)$.

f is bounded ie. $|f(z)| \leq M \quad \forall z \in D'(0,R)$.

By estimation $$|c_k| \leq \frac{1}{2\pi}\max_{z\in\Gamma}\left|\frac{f(z)}{z^{k+1}}\right| L(\Gamma)$$

$$\leq r\frac{M}{r^{k+1}} = \frac{M}{r^k}$$

r is arbitrary $\Rightarrow$ for $k<0$, $c_k = 0$.

ie. removable singularity.

(c) $f(z) = \dfrac{1}{\sin z}$ has smallest +ve ~~root~~ at $z = \pi$

$\Rightarrow$ RoC $= \pi$.

$$\frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{6} + \dots} = \frac{1}{z} + \frac{z}{6} + \dots$$

principle part! z^{-1}.

4. CIF :
(a)

$$\cancel{\oint_{\Gamma} f(z)\,dz = 2\pi i \sum_{i=1}^{n} \mathrm{Res}(f, z_i)}$$

~~where f(z) has singularities $z_i, \ldots, z_n$~~

$$f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - z_0}\,dz$$

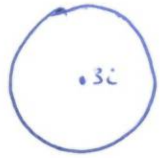
f holomorphic in simply connected domain Ω.

Γ simple closed contour in Ω enclosing z_0, then...

(b) (i) $I_1 = \int_{\Gamma_1} \frac{z}{z^2+4}\,dz$

$\frac{z}{z^2+4} = f$

f has roots at $z = \pm 2i$.

$\Gamma_1 =$ 

$-2i$ is not inside the circle

$$\Rightarrow I_1 = 2\pi i \,\mathrm{Res}(f, 2i)$$

$$= 2\pi i \lim_{z \to 2i} \frac{z(z-2i)}{z^2+4} = 2\pi i \lim_{z \to 2i} \frac{z}{z+2i} = 2\pi i \left(\frac{1}{2}\right) = \pi i$$

(ii) Both roots inside $\Rightarrow I_2 = 2\pi i [\mathrm{Res}(f, 2i) + \mathrm{Res}(f, -2i)]$

$$= 2\pi i \left[\frac{1}{2} + \lim_{z \to -2i} \frac{z}{z-2i}\right] = 2\pi i \left[\frac{1}{2} + \frac{1}{2}\right] = 2\pi i.$$

(c) $h(z) = \sin(\mathrm{Im}\, z)$.

$h(z) = \sin y + i \cdot 0$

$u_y = \cos y \neq -v_x = 0$.

for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

5 (a) A f^n f with isolated singularity z_0 has Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$.

Pole: if $\exists M \in \mathbb{N}$ s.t. $\forall k < -M$, $c_k = 0$; $c_{-M} \neq 0$,
f is said to have a pole of order M

Ess: If $\nexists M$ s.t. $\forall k < M$, $c_k = 0$,
f is said to have an essential singularity.

Rem: If $\forall k < 0$, $c_k = 0$
f is said to have a removable singularity.

(b) $f(z) = \frac{z}{z^2-1} = \frac{z}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1}$

$Az - A + Bz + B = z$
$\rightarrow B = A$
$A = \frac{1}{2}$

$= \frac{1}{2(z+1)} + \frac{1}{2(z-1)}$
$= g(z) + h(z)$

(i) $0 < |z-1| < 2$

$g(z) = \frac{1}{2(z+1)} = \frac{1}{2(z-1)+4} = \frac{1}{4}\left[\frac{1}{1+\frac{1}{2}(z-1)}\right]$

$= \cancel{\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{-2}{z-1}\right)^n}$

$= \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{-(z-1)}{2}\right)^n$

$= \sum_{n=0}^{\infty} (-1)^n (z-1)^n \left(\frac{1}{2^{n+2}}\right)$

$\Rightarrow f(z) = \frac{1}{2(z-1)} + \sum_{n=0}^{\infty} (-1)^n (z-1)^n \left(\frac{1}{2^{n+2}}\right)$

I prefer the substitution approach.

(ii) $w = z+1$
$z = w-1$

$|w| > 2$

$$f(v) = \frac{w-1}{(w-1)^2-1} = \frac{A}{w} + \frac{B}{w-2} \Rightarrow Aw-2A \ldots$$

$A + B = \ldots$
$2A = 1$

$$= \frac{1}{2w} + \frac{1}{2(w-2)}$$

$$= \frac{1}{2w\left(1-\frac{2}{w}\right)}$$

$$= \frac{1}{2w} \sum_{n=0}^{\infty} \left(\frac{2}{w}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{2^{n-1}}{w^{n+1}}$$

$$\rightarrow f(w) = \frac{1}{2w} + \sum_{n=0}^{\infty} \frac{2^{n-1}}{w^{n+1}}$$

$$= \frac{1}{2w} + \frac{1}{2w} + \sum_{n=1}^{\infty} \frac{2^{n-1}}{w^{n+1}}$$

$$= \frac{1}{w} + \sum_{n=1}^{\infty} \frac{2^{n-1}}{w^{n+1}}$$

$$= \frac{1}{z+1} + \sum_{n=1}^{\infty} \frac{2^{n-1}}{(z+1)^{n+1}}$$

(iii) $|z| > 1$:

$$f(z) = \frac{z}{z^2-1} \cancel{= \frac{-z}{1-z^2}} = \cancel{-z \sum_{n=0}^{\infty} z^{2n}} \cancel{= -\sum_{n=0}^{\infty} z^{2n+1}}$$

$$= \frac{z}{z^2\left(1-\frac{1}{z^2}\right)} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z^2}\right)^n$$

$$= \sum_{n=0}^{\infty} z^{-2n-1}$$

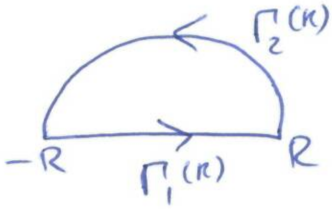
(a) $\text{Res}(f, z_0) = c_{-1}$ where f has Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$.

(b) $\cancel{\text{Res}(g,0) = \lim_{z \to 0} \frac{\cosh z}{z}}$

0 is a pole of order 2

$\Rightarrow \text{Res}(g,0) = \lim_{z\to 0} \frac{d}{dz}\left(\cosh z\right) = 0$.

(c) $\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4}\, dx$



$\Gamma(R) = \Gamma_1(R) + \Gamma_2(R)$

$\Gamma_1(R) = \{z : \text{Im} z = 0, |\text{Re} z| \le R\}$

$\Gamma_2(R) = \{Re^{i\theta} : \theta \in [0, \pi]\}$.

x^2+4 has roots at $x = \pm 2i$

Only $2i$ is in this contour

so evaluate

$$\int_{\Gamma} \underbrace{\frac{e^{i3z}}{z^2+4}}_{f} dz$$

$\text{Res}(f, 2i) = \lim_{z \to 2i} \frac{e^{i3z}}{z+2i} = \frac{e^{-6}}{4i}$

$\Rightarrow 2\pi i \,\text{Res}(f, 2i) = \frac{\pi}{2} e^{-6}$

$\Rightarrow \int = \text{Re}\left(\frac{\pi}{2}e^{-6}\right) = \frac{\pi}{2}e^{-6}$.

1 (a) (i) $\frac{df}{dz}(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$

(ii) f^n is hol on Ω if it's diff^{le} at every pt in Ω.

(b) f is holomorphic on $D(0,1)$,
need to prove $g(z) = \overline{f(\bar{z})}$

$$\frac{g(z_0+h) - g(z_0)}{h} = \frac{\overline{f(\overline{z_0+h})} - \overline{f(\bar{z_0})}}{h}$$

$$= \overline{\left[\frac{f(\overline{z_0+h}) - f(\bar{z_0})}{\bar{h}}\right]} \qquad \overline{\left[\frac{f(w+H) - f(w)}{H}\right]}$$

Let $\bar{z_0} = w$ $\quad \lim_{H\to 0} \quad = \overline{f'(w)} = \overline{f'(\bar{z})}$.
$\bar{h} = H$

(c) Trivial?

2 (a) $u_x = v_y \qquad u_y = -v_x$

(b) $f(z) = u + iv$. $\qquad |f(z)| = \sqrt{u^2+v^2} = \text{const} = a$.
$\Rightarrow \cancel{z} \rightarrow u^2+v^2 = a^2$

$$\Rightarrow \begin{array}{l} 2uu_x + 2vv_x = 0 \\ 2uu_y + 2vv_y = 0 \end{array} \qquad \begin{array}{ll} \times u: & u^2u_x + uvv_x = 0 \\ \times v: & uvu_y + v^2v_y = 0 \\ + & \overline{\qquad\qquad\qquad} \\ & \underbrace{(u^2+v^2)}_{c^2}u_x = 0 \end{array}$$

$\Rightarrow u_x = \text{const}$. by similar, $u_x = u_y = v_x = v_y = \text{const}$.
$\Rightarrow f$ const.

(c) (i) $\sin z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} (-1)^k$

$$\cos z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} (-1)^k$$

(ii) $\cos z = \cos(z - z_0 + z_0)$

$$= \cos(z-z_0)\cos z_0 - \sin(z-z_0)\sin z_0$$

$$= \cos z_0 \sum_{k=0}^{\infty} \frac{(z-z_0)^{2k}}{2k!}(-1)^k - \sin z_0 \sum_{k=0}^{\infty} \frac{(z-z_0)^{2k+1}}{(2k+1)!}(-1)^k$$

(iii) Diff^n for power series,
the derivative of a series is another series with same RoC
and terms obtained by term-by-term diff^n.

$\rightarrow$ simple [illegible] ch.

(iv) Let $f(z) = \sin^2 z + \cos^2 z$
$f'(z) = 2\sin z\cos z - 2\cos z \sin z = 0$

$\rightarrow f(z) = \text{const}$.
$f(0) = 1 \quad \rightarrow \quad f(z) = 1. \quad \forall z.$

3 (a) Suppose $\left.\begin{array}{l} f = u + iv \\ g = v + iu \end{array}\right\}$ are analytic in D.

$\Rightarrow \left.\begin{array}{l} u_x = v_y \\ -u_y = v_x \end{array}\right\}$ for f for g $\left\{\begin{array}{l} v_x = u_y \\ -v_y = u_x \end{array}\right.$

$\Rightarrow$

$u_x = u_y = v_x = v_y = 0$.

$\Rightarrow u, v$ const f^ns.

(b). $c_k = \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \frac{f(w)}{w^{k+1}}\, dw$ Γ positively oriented circular contour centred at $z_0 = 0$ of radius $r \in (0, R)$.

Let $|f(\cancel{w}z)| \le M \quad \forall z \in D(0, R)$.

By estimation, $|c_k| \le \dfrac{1}{2\pi} \max\limits_{z \in \Gamma} \left| \dfrac{f(z)}{z^{k+1}} \right| L(\Gamma) \le \dfrac{M}{r^k}$

r is arbitrary $\Rightarrow$ for $k < 0$, $c_k = 0$

$\Rightarrow$ Laurent expansion contains only the terms with non-negative powers of z $\Rightarrow$ holomorphic.

(c) $f(z) = \dfrac{1}{\sin z}$ has smallest positive roots at π $\Rightarrow R = \pi$

Using standard Taylor series $\dfrac{1}{\sin z} = \dfrac{1}{z - \frac{z^3}{6} + \frac{z^5}{5!} \cdots}$

$= \dfrac{1}{z} + \dfrac{z}{6} + \cdots$

$\Downarrow$

pricipal part of L. exp. is z^{-1}.

4. $\oint f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}\,dz$

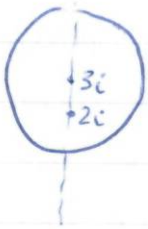
f holomorphic in simply connected domain Ω.

Γ simple closed contour in Ω, enclosing $z_0 \in U$, then

(a)

(b) $z^2+4 = \cancel{0}$ has roots $\pm 2i$. ~~$-2x$~~

$= (z-2i)(z+2i)$

(i) 

$-2i$ is not in the circle

$$\Rightarrow \int_{\Gamma_1} \frac{z}{z^2+4}\,dz = 2\pi i \lim_{z\to 2i} \frac{z}{z+2i} = \cancel{\tfrac{1}{2}\,4i\,2i}\ 2\pi i\,\frac{2i}{4i} = \pi i$$

(ii) Both are in the circle

$$\Rightarrow \int_{\Gamma_2} \underbrace{\frac{z}{z^2+4}}_{f}\,dz = 2\pi i\left[\mathrm{Res}(f,2i) + \mathrm{Res}(f,-2i)\right]$$

$$= 2\pi i\left[\cancel{\ }\ \frac{1}{2} + \frac{-2i}{-}\ \text{ch}\right] = 2\pi i.$$

(c) $u_x = v_y$

$u_y = -v_x$

$h(x) = \sin(\mathrm{Im}\,z) = \sin y.$

Let $h(x) = u(x,y) + iv(x,y)$

$= \sin y + 0.$

Then $u_y \neq -v_x \Rightarrow$ not diff^{le}.

5 (a) ~~Pole if~~

if there is an isolated singularity, f has L. expansion

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k$$

$M>0$.

Pole if $\forall k < -M$, $c_k = 0$ and $c_{-M} \neq 0$ (pole of order M)

Ess if $\nexists\, l$ s.t. $\forall k < l$, $c_k = 0$

Rem if $\forall k < 0$, $c_k = 0$.

(b) $f(z) = \dfrac{z}{z^2-1} = \dfrac{A}{z+1} + \dfrac{B}{z-1}$ ⟶ $A+B=1$, $-A+B=0$

$$= \frac{1}{2(z+1)} + \frac{1}{2(z-1)}$$

$$= g(z) + h(z).$$

(i) $0 < |z-1| < 2$:

$$g(z) = \frac{1}{2(z+1)} = \frac{1}{2(z-1)+4} = \frac{1}{4}\,\frac{1}{\frac{1}{2}(z-1)+1}$$

$$= \frac{1}{4}\sum_{k=0}^{\infty} (-1)^k \frac{1}{2^k}(z-1)^k$$

$$= \frac{1}{4} + \sum_{k=1} (-1)^k \frac{1}{2^{k+2}}(z-1)^k$$

$$\Rightarrow f(z) = \frac{1}{2(z-1)} + \frac{1}{4} + \sum_{k=1}^{\infty} (-1)^k \frac{1}{2^{k+2}}(z-1)^k$$

(ii) $$\frac{z}{z^2-1} = \frac{1}{2(z+1)} + \frac{1}{2(z-1)}$$
$$= g(z) + h(z).$$

$$\cancel{h(z) = \frac{1}{2(z+1)-4} = \frac{1}{4\left[\frac{1}{2}(z+1)-1\right]} = -\frac{1}{4}\,\frac{1}{1-\frac{1}{2}(z+1)}}$$
$$\cancel{= -\frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{z+1}{2}\right)^k}$$

$$h(z) = \frac{1}{2(z-1)} = \frac{1}{2(z+1)-4} = \frac{1}{z+1}\cdot\frac{1}{2-\frac{4}{z+1}}$$
$$= \frac{1}{2(z+1)}\cdot\frac{1}{1-\frac{2}{z+1}}$$
$$= \frac{1}{2(z+1)}\sum_{k=0}^{\infty}\left(\frac{2}{z+1}\right)^k$$
$$= \cancel{\tfrac{1}{2}}\sum_{k=0}^{\infty}\frac{2^{k-1}}{(z+1)^{k+1}}$$

$$\Rightarrow f(z) = \frac{1}{2(z+1)} + \sum_{k=0}^{\infty}\frac{2^{k-1}}{(z+1)^{k+1}}$$
$$= \frac{1}{2(z+1)} + \frac{1}{2(z+1)} + \sum_{k=1}^{\infty}\frac{2^{k-1}}{(z+1)^{k+1}}$$
$$= \frac{1}{z+1} + \sum_{k=1}^{\infty}\frac{2^{k-1}}{(z+1)^{k+1}}$$

(iii) $$f(z) = \frac{z}{z^2-1} = \frac{z}{z^2(1-\frac{1}{z^2})} = \frac{1}{z(1-\frac{1}{z^2})} = \frac{1}{z}\sum_{k=0}\left(\frac{1}{z^2}\right)^k$$

$$= \sum_{k=0} z^{-2k-1}$$

6 (a) ~~$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$~~

Let p be an isolated singularity of f ~~at~~, then

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-p)^k \quad \text{is the L- exp.}$$

Then $\mathrm{Res}(f, p) = c_{-1}$.

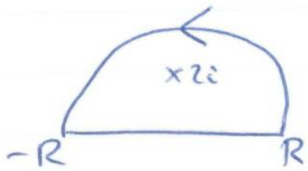
(b) $g(z) = \dfrac{\cosh z}{z^2}$. Pole of order 2 here.

~~CRT. CRT.~~

$$\mathrm{Res}(g, 0) = \lim_{z \to 0} \frac{d}{dz}(\cosh z) = 0.$$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4}\,dx$ has poles at $x = \pm 2i$

~~Integrate~~



$\Gamma_1 = \{z : \mathrm{Im}\, z = 0,\ |z| \leq R\}$

$\Gamma_2 = \{z = Re^{i\theta} : \theta \in [0, \pi]\}$

By CRT, $\displaystyle\int_\Gamma \frac{e^{i3z}}{z^2+4}\,dz = 2\pi i \lim_{z \to 2i} \frac{e^{i3z}}{(z+2i)} = \frac{e^{-6}}{4i}$

$$= \frac{\pi}{2} e^{-6}.$$

By Jordan's Lemma, with $g(z) = e^{i3z} f(z)$, $f(z) = \dfrac{1}{z^2+4}$.

As $|f(z)| \leq \dfrac{1}{R^2+4}$ on Γ, $M(R) = \dfrac{1}{R^2+4}$

and $\lim_{R \to \infty} M(R) = 0$

$\Rightarrow \displaystyle\lim_{R \to \infty} \int_\Gamma g(z)\,dz = \int_{-\infty}^{\infty} \frac{e^{3ix}}{x^2+4}\,dx$. Real parts: $I = \frac{\pi}{2} e^{-6}$